THE USE OF COPULAS AND MPP-BASED DIMENSION REDUCTION METHOD (DRM) TO ASSESS AND MITIGATE ENGINEERING RISK IN THE ARMY GROUND VEHICLE FLEET

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ABSTRACT

In reliability based design optimization (RBDO) problems with correlated input variables, a joint cumulative distribution function (CDF) needs to be obtained to transform, using the Rosenblatt transformation, the correlated input variables into independent standard Gaussian variables for the inverse reliability analysis. However, a true joint CDF requires infinite number of test data to be obtained, so in this paper, a copula is used, which models a joint CDF only using marginal CDFs and limited data. Then, the inverse reliability analysis can be carried out using the joint CDF modeled by the copula and the first order reliability method (FORM), which has been commonly used in the inverse reliability analysis. However, because of the nonlinear Rosenblatt transformation, the FORM may yield inaccurate reliability analysis results. To resolve the problem, this paper proposes to use the most probable point (MPP)-based dimension reduction method (DRM) for more accurate inverse reliability analysis and RBDO. As an example of the proposed method, an RBDO study of an M1A1 Abrams tank roadarm is carried out.

1. INTRODUCTION

In many RBDO problems of automotive engineering, input random variables, such as material fatigue properties, are correlated (Socie, 2003; Annis, 2004; Efstratios et al., 2004). For the RBDO problem with the correlated input variables, the joint CDF of the input variables should be available to transform the correlated input variables into the independent standard Gaussian variables, by using the Rosenblatt transformation (Rosenblatt, 1952) to carry out the inverse reliability analysis. However, in industrial applications, often only the marginal CDFs and limited paired sampled data are available using experimental testing, which makes it very difficult to obtain the input joint CDF. In this paper, a copula, which links the joint CDF and marginal CDFs, is used to model the joint CDF. Since the copula only requires marginal CDFs and correlation parameters, which are often available in industrial applications, to model the joint CDF, the joint CDF can be readily generated. Thus, it is valuable to use the copula for modeling the joint CDFs in practical applications with correlated input variables.

Once the joint CDF is modeled using the copula, the Rosenblatt transformation can be utilized to transform the original random variables into the independent standard Gaussian variables for the inverse reliability analysis. For the inverse reliability analysis, the FORM has been widely used due to its simplicity of the probability of failure calculation. However, if the input variables have a non-Gaussian joint CDF modeled by a non-Gaussian copula, which often occurs in automotive engineering applications (Pham, 2006), the Rosenblatt transformation becomes nonlinear, which can significantly affect the nonlinearity of the transformed constraints. In this case, the FORM may not yield accurate inverse reliability analysis results since the FORM uses a linear approximation of the constraint to estimate the probability of failure. To obtain more accurate inverse reliability analysis and RBDO results, the MPP-based DRM (Lee et al., 2008) is introduced in this paper.

An M1A1 Abrams tank roadarm problem (Lee et al., 2008) is used as an example of RBDO using the copula and MPP-based DRM. The example shows that the weight of the roadarm is significantly reduced using the copula instead of assuming that inputs are independent, and, the weight is further reduced using the MPP-based DRM.

2. RBDO FORMULATION

The RBDO problem can be formulated to

$$
\begin{aligned}
\text{min.}\quad & \text{Cost}(\mathbf{d}) \\
\text{s.t.}\quad & P(G_i(\mathbf{X})>0)\le P_{F_i}^{Tar},\ i=1,\cdots,nc \\
& \mathbf{d}=\mu(\mathbf{X}),\ \mathbf{d}_{\mathrm{L}}\le\mathbf{d}\le\mathbf{d}_{\mathrm{U}},\ \ \mathbf{d}\in R^{ndv}\ \text{and}\ \mathbf{X}\in R^{n}
\end{aligned}
\tag{1}
$$

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where **X** is the vector of random variables; **d** is the vector of design variables; $G_i(\mathbf{X})$ represents the i[th] constraint functions; $P_{F_i}^{Tar}$ is the given target probability of failure for the i[th] constraint; and *nc*, *ndv*, and *n* are the number of probabilistic constraints, number of design variables, and number of random variables, respectively. The probability of failure in Eq. (1) is estimated by a multi-dimensional integral of the joint PDF of the input variables over the failure region as

$$P\left(G_i\left(\mathbf{X}\right)>0\right)=\int_{G_i(\mathbf{X})>0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x},\ i=1,\cdots,nc \tag{2}$$

where **x** is the realization of the random vector **X** and $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (PDF) of **X**. However, since it is very difficult to compute the multi-dimensional integral, approximation methods such as the FORM or the second order reliability method (SORM) are used. The FORM often provides adequate accuracy and is much easier to use than the SORM, and hence it has been commonly used in RBDO.

Using a performance measure approach (PMA+) (Youn et al., 2005), the i[th] constraint in Eq. (1) can be rewritten as

$$P[G_i(\mathbf{X})>0]-P_{F_i}^{\mathrm{Tar}}\le 0 \Rightarrow G_i\,(\mathbf{x}^*)\le 0 \tag{3}$$

where $G_i\,(\mathbf{x}^*)$ is the i[th] constraint function evaluated at the most probable point (MPP), $\mathbf{x}^*$, in X-space, which can be obtained by solving the following optimization problem:

$$\begin{aligned} &\text{max.} && g_i(\mathbf{u}) \\ &\text{s.t.} && \|\mathbf{u}\|=\beta_{t_i} \end{aligned} \tag{4}$$

where $g_i(\mathbf{u})$ is the i[th] constraint function that is transformed from the original space (X-space) into the standard Gaussian space (U-space), i.e., $g_i(\mathbf{u})\equiv G_i(\mathbf{x}(\mathbf{u}))=G_i(\mathbf{x})$ and β_{t_i} is the target reliability index such that $P_{F_i}^{\mathrm{Tar}}=\Phi\left(-\beta_{t_i}\right)$ using the FORM. If the constraint function at the MPP is less than or equal to zero, then the i[th] constraint in Eq. (1) is satisfied for the given target reliability. Thus, Eq. (1) can be rewritten using PMA+ as

$$\begin{aligned} &\text{min.} && \mathrm{Cost}(\mathbf{d}) \\ &\text{s.t.} && G_i(\mathbf{x}^*)\le 0,\ i=1,\cdots,nc \\ & && \mathbf{d}^{\mathrm{L}}\le\mathbf{d}\le\mathbf{d}^{\mathrm{U}},\quad \mathbf{d}\in\mathrm{R}^{ndv}\ \text{and}\ \mathbf{X}\in\mathrm{R}^{n} \end{aligned} \tag{5}$$

3. MODELING OF JOINT CDF USING COPULA

As mentioned earlier, if the input variables are correlated, it is often too difficult to obtain the true joint CDF in practical industrial applications with only limited experimental data. In this paper, a copula is used to model the joint CDF using marginal CDFs and correlation measures that are calculated from the experimental data. The definition of copula and the correlation measures associated with copulas are explained in this section.

3.1 Definition of Copula

Copula is originated from a Latin word for “link” or “tie” that connects two different things. In statistics, the definition of copula is stated by Nelson (Nelson, 1999): “Copulas are functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions. Alternatively, copulas are multivariate distribution functions whose one-dimensional margins are uniform on the interval [0, 1].”

According to Sklar’s theorem (Nelson, 1999), if the random variables have a joint CDF $F_{X_1\cdots X_n}\left(x_1,...,x_n\right)$ with marginal distributions, $F_{X_1}\left(x_1\right),...,F_{X_n}\left(x_n\right)$, then there exists an n-dimensional copula *C* such that

$$F_{X_1,...,X_n}\left(x_1,...,x_n\right)=C\left(F_{X_1}\left(x_1\right),...,F_{X_n}\left(x_n\right)\middle|\boldsymbol{\theta}\right) \tag{6}$$

where $\boldsymbol{\theta}$ is the matrix of the correlation parameters of $x_1,...,x_n$. If marginal distributions are all continuous, then the copula *C* is unique. Conversely, if *C* is an *n*-dimensional copula and $F_{X_1}\left(x_1\right),...,F_{X_n}\left(x_n\right)$ are the marginal distributions, then $F_{X_1\cdots X_n}\left(x_1,...,x_n\right)$ is the joint CDF (Nelson, 1999). By taking the derivative of Eq. (6), the joint PDF $f_{X_1,\cdots,X_n}\left(x_1,\cdots,x_n\right)$ is obtained as

$$f_{X_1,\cdots,X_n}\left(x_1,\cdots,x_n\right)=c\left(F_{X_1}\left(x_1\right),\cdots,F_{X_n}\left(x_n\right)\middle|\boldsymbol{\theta}\right)\prod_{i=1}^{n}f_{X_i}\left(x_i\right) \tag{7}$$

where $c\left(u_1,\cdots,u_n\right)\equiv\dfrac{\partial^n C\left(u_1,\cdots,u_n\right)}{\partial u_1\cdots\partial u_n}$ with $u_i=F_{X_i}\left(x_i\right)$, and $f_{X_i}\left(x_i\right)$ is the marginal PDF for $i=1,\cdots,n$. A copula only requires marginal CDFs and correlation parameters to model a joint CDF, so the joint CDF can be readily obtained from limited data. In addition, since the copula decouples marginal CDFs from the joint CDF, the joint CDF modeled by the copula can be expressed in terms of any type of marginal CDF. That is, having marginal Gaussian CDFs does not mean that the joint CDF is Gaussian. Thus, it is desirable to be able to model the joint CDF of correlated input variables with mixed types of marginal CDFs, which can often occur in industrial applications. To model the joint CDF using the copula, the correlation parameters need to be obtained from experimental data as seen in Eqs. (6) and (7). Since various types of copulas have their own correlation parameters, it is desirable to have a common correlation measure to obtain the correlation parameters from the experimental data.

3.2 Correlation Measures

To measure the correlation between two random variables, Pearson's rho and Kendall's tau can be used. Pearson's rho was first discovered by Bravais (Bravais, 1846), and was developed by Pearson (Pearson, 1896). Pearson's rho indicates the degree of linear relationship between two random variables as

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \tag{8}$$

where σ_X and σ_Y are standard deviations of X and Y, respectively, and $Cov(X,Y)$ is the covariance between X and Y. Since Pearson's rho only indicates the linear relationship between two random variables, it is valid only when the joint CDF is Gaussian. Pearson's rho also can be used as correlation measure in the joint CDF modeled by Gaussian copula, because the Gaussian copula is originated from a joint Gaussian CDF. If the marginal CDFs are Gaussian, then the joint CDF modeled by the Gaussian copula is the joint Gaussian CDF. The Gaussian copula allows generating a joint Gaussian CDF with non- marginal Gaussian CDFs as

$$C_{\Phi}\left(u_1,\cdots,u_n \middle| \mathbf{P'}\right) = \Phi_{\mathbf{P'}}\left(\Phi^{-1}(u_1),\cdots,\Phi^{-1}(u_n) \middle| \mathbf{P'}\right), \quad \mathbf{u} \in I^n \tag{9}$$

where $u_i = F_{X_i}(x_i)$ is the marginal CDF of X_i for $i = 1,\cdots,n$, $\mathbf{P'}$ is the covariance matrix consisting of correlation coefficients, Pearson's rho, between correlated input variables. $\Phi(\cdot)$ represents the marginal standard Gaussian CDF and $\Phi_{\mathbf{P'}}(\cdot)$ is the joint Gaussian CDF defined as

$$\Phi_{\mathbf{P'}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} (\mathbf{P'})^{\mathbf{-1}} (\mathbf{x}-\boldsymbol{\mu})\right] \tag{10}$$

for $\mathbf{x} = [x_1,\cdots,x_n]^{\mathrm{T}}$ with a mean vector $\boldsymbol{\mu} = [\mu_1,\cdots,\mu_n]^{\mathrm{T}}$. However, Pearson's rho cannot be a good measure for a nonlinear relationship between two random variables, which often occurs in practical engineering applications. If the given data follows a joint non-Gaussian CDF modeled by a non-Gaussian copula, another correlation measure is necessary.

Unlike Pearson's rho, Kendall's tau does not require the assumption that the relationship between two random variables is linear. Since the Kendall's tau measures the correspondence of rankings between correlated random variable, it is called a rank correlation coefficient. The Kendall's tau was first introduced by Kendall (Kendall, 1938) and is defined as

$$\tau = 4\int\int_{I^2} C(u,v|\theta)\, dC(u,v) - 1 \tag{11}$$

where $I^n = I \times I \left(I = [0,1]\right)$ and Eq. (11) is the population version of Kendall's tau. The sample version of Kendall's tau is

$$t = \frac{c-d}{c+d} = (c-d)\Big/\binom{ns}{2} \tag{12}$$

where c represents the number of concordant pairs, d is the number of discordant pairs, and ns is the number of samples. Using the estimated Kendall's tau, the correlation parameter of the copula, θ, can be calculated because Kendall's tau can be expressed as a function of the correlation parameter as shown in Eq. (11). The explicit functions of Eq. (12) for some copulas are presented in reference (Huad et al., 2006) .

Consider a non-Gaussian copula, which uses a rank correlation coefficient such as Kendall's tau as the correlation measures. Unlike the Gaussian copula, the Archimedean copula is constructed in a completely different way. An important component of constructing Archimedean copula is a generator function φ_θ with a correlation parameter θ. If φ_θ is a continuous and strictly decreasing function from $[0,1]$ to $[0,\infty)$ such that $\varphi_\theta(0) = \infty$ and $\varphi_\theta(1) = 0$ and the inverse φ_θ^{-1} is completely monotonic on $[0,\infty)$, then the Archimedean copula can be defined as (Nelson, 1999)

$$C\left(u_1,\cdots,u_n \middle| \theta\right) = \varphi_\theta^{-1}\left[\varphi_\theta(u_1) + \cdots + \varphi_\theta(u_n)\right] \tag{13}$$

for $n \geq 2$. Each Archimedean copula has a corresponding unique generator function φ_θ, which provides a multivariate copula as shown in Eq. (13). Once the generator function is provided, the Kendall's tau can be obtained as

$$\tau = 1 + 4\int_0^1 \frac{\varphi_\theta(t)}{\varphi_\theta'(t)} dt \tag{14}$$

Using Eq. (14), the correlation parameter θ can be expressed in terms of Kendall's tau. More detailed information on Kendall's tau is presented in reference (Kendall, 1938).

Table 1. Copula Functions and Their Parameter Domain

Copula	$C(u,v\|\theta)$	$\theta \in \Omega$
Clayton	$\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}$	$(0,\infty)$
AMH	$uv/\left[1-\theta(1-u)(1-v)\right]$	$[-1,1)$
Gumbel	$\exp\left\{-\left[(-\ln u)^{\theta}+(-\ln v)^{\theta}\right]^{1/\theta}\right\}$	$[1,\infty)$
Frank	$-\frac{1}{\theta}\ln\left[1+\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)/\left(e^{-\theta}-1\right)\right]$	$(-\infty,\infty)$
A12	$\left\{1+\left[\left(u^{-1}-1\right)^{\theta}+\left(v^{-1}-1\right)^{\theta}\right]^{1/\theta}\right\}^{-1}$	$[1,\infty)$
A14	$\left\{1+\left[\left(u^{-1/\theta}-1\right)^{\theta}+\left(v^{-1/\theta}-1\right)^{\theta}\right]^{1/\theta}\right\}^{-\theta}$	$[1,\infty)$

FGM	$uv + \theta uv(1-u)(1-v)$	$[-1,1]$
Gaussian	$\int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{\exp\left(\frac{2\theta sw - s^2 - w^2}{2(1-\theta^2)}\right)}{2\pi\sqrt{1-\theta^2}} ds dw$	$[-1,1]$

The Archimedean copula can be used for a multivariate CDF. But it is hard to expand to an *n*-dimensional copula because, as shown in Eq. (14), it has one generator function, and thus has the same correlation parameter even if *n* variables are correlated with different correlation coefficients. Hence, most copula applications consider bivariate data only, so does this paper.

Including the Gaussian copula and Archimedean copula, there exist various kinds of copulas as listed in Table 1. Thus, selecting an appropriate copula is necessary to correctly model a joint CDF based on the given experimental data. The identification of the true copula is addressed in detail in references (Huad et al., 2006; Noh et al., 2008).

4. MPP-BASED DIMENSION REDUCTION METHOD

The MPP-based reliability analysis such as the FORM (Hasofer and Lind, 1974; Tu and Choi, 1999) and the SORM (Breitung, 1984; Hohenbichler and Rackwitz, 1988) has been commonly used for reliability or inverse reliability assessment. However, when the constraint function is nonlinear or multi-dimensional, the reliability analysis using the FORM could be significantly erroneous because the FORM cannot handle the complexity of nonlinear or multi-dimensional functions. Inverse reliability analysis using the SORM may be accurate, but the second-order derivatives required for the SORM are very difficult and expensive to obtain in industrial applications. On the other hand, the MPP-based DRM achieves both the efficiency of the FORM and the accuracy of the SORM (Lee et al., 2008).

The DRM was developed to accurately and efficiently approximate a multi-dimensional integral (Rahman and Xu, 2004). There are several DRMs depending on the level of dimension reduction: univariate dimension reduction, bivariate dimension reduction, and multivariate dimension reduction. In this paper, the univariate DRM is used for calculating probability of failure due to its simplicity and efficiency. The univariate DRM is carried out by decomposing an *n*-dimensional constraint function *G*(X) into the sum of one-dimensional functions at the MPP as (Rahman and Wei, 2006)

$$\begin{aligned} G(\mathbf{X}) &\cong \hat{G}(\mathbf{X}) \\ &\equiv \sum_{i=1}^{n} G(x_1^*, \cdots, x_{i-1}^*, X_i, x_{i+1}^*, \cdots, x_n^*) - (n-1)G(\mathbf{x}^*) \end{aligned} \tag{15}$$

where $\mathbf{x}^* = \{x_1^*, x_2^*, \cdots, x_n^*\}^{\mathrm{T}}$ is the FORM-based MPP obtained from Eq. (4) and *n* is the number of random variables. In the inverse reliability analysis, since the probability of failure cannot be directly calculated in U-space, a constraint shift in a rotated standard Gaussian space (V-space) needs to be defined as

$$\tilde{G}^s(\mathbf{v}) \equiv \tilde{G}(\mathbf{v}) - \tilde{G}(\mathbf{v}^*) \tag{16}$$

where $\mathbf{v}^* = \{0, \cdots, 0, \beta\}^{\mathrm{T}}$ is the MPP in V-space and $\tilde{G}(\mathbf{v}) \equiv G(\mathbf{x}(\mathbf{v}))$. Then, using the shifted constraint function, the probability of failure using the MPP-based DRM is calculated as (Lee et al., 2008)

$$P_F^{\mathrm{DRM}} = \frac{\prod_{i=1}^{n-1} \int_{-\infty}^{\infty} \Phi(-\beta + \frac{\tilde{G}_i^s(v_i)}{b_1}) \phi(v_i) dv_i}{\Phi(-\beta)^{n-2}} \tag{17}$$

where $\tilde{G}_i^s(v_i) \equiv \tilde{G}^s(0, \cdots, 0, v_i, 0, \cdots, \beta)$ is a function of v_i only and $b_1 = \left\| \frac{\partial g(\mathbf{u}^*)}{\partial \mathbf{u}} \right\|$.

Equation (17) can be further approximated as using the moment-based integration rule (Xu and Rahman, 2003)

$$P_F^{\mathrm{DRM}} = \frac{\prod_{i=1}^{n-1} \sum_{j=1}^{N} w_j \Phi\left(-\beta + \frac{\tilde{G}_i^s(v_i^j)}{b_1}\right)}{\Phi(-\beta)^{n-2}} \tag{18}$$

where v_i^j represents the j^{th} quadrature point for v_i, w_j denote weights, and *N* is the number of quadrature points. The quadrature points and weights for the standard Gaussian random variables v_i are shown in Table 2.

Table 2. Gaussian Quadrature Points and Weights

N	Quadrature Points	Weights
1	0.0	1.0
3	$\pm\sqrt{3}$	0.166667
	0.0	0.666667
5	±2.856970	0.011257
	±1.355626	0.222076
	0.0	0.533333

Using P_F^{DRM} calculated from Eq. (18), the corresponding reliability index β_{DRM} can be defined as

$$\beta_{\mathrm{DRM}} = -\Phi^{-1}(P_F^{\mathrm{DRM}}), \tag{19}$$

which is not the same as the target reliability index $\beta_t = -\Phi^{-1}(P_F^{\mathrm{Tar}})$ because the nonlinearity of the constraint function is reflected in the calculation of P_F^{DRM}. Hence, using β_{DRM}, a new updated reliability index β_{up} can be defined as

$$\beta_{\mathrm{up}} \equiv \beta_{\mathrm{cur}} + \Delta\beta = \beta_{\mathrm{cur}} + (\beta_t - \beta_{\mathrm{DRM}}) \tag{20}$$

where β_{cur} is the current reliability index. The recursive form of the Eq. (20) is

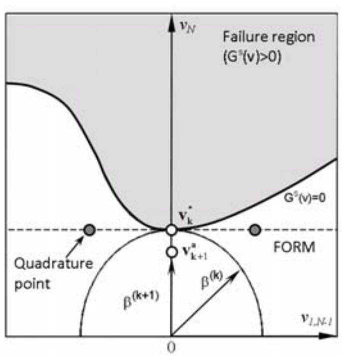
$$\beta^{(k+1)} \equiv \beta^{(k)} + \Delta\beta = \beta^{(k)} + (\beta_t - \beta_{\mathrm{DRM}}) \tag{21}$$

where $\beta^{(0)} = \beta_t$ at the initial step.

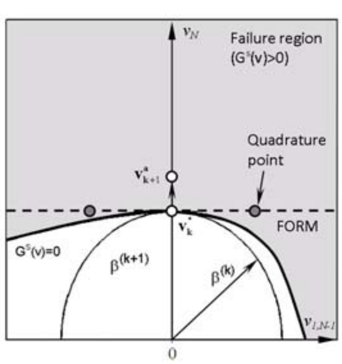
Using this updated reliability index, the updated MPP can be found by either using an iterative MPP search or using an approximation. If an iterative MPP search with the updated reliability index is used, the procedure will be computationally expensive. Accordingly, to improve the efficiency of the optimization, the updated MPP can be approximated as (Ba-abbad et al., 2006)

$$\mathbf{u}^{\mathbf{a}}_{\mathbf{k+1}} \cong \frac{\beta^{(k+1)}}{\beta^{(k)}}\mathbf{u}^{*}_{\mathrm{k}} \text{ or } \mathbf{v}^{\mathbf{a}}_{\mathbf{k+1}} \cong \frac{\beta^{(k+1)}}{\beta^{(k)}}\mathbf{v}^{*}_{\mathrm{k}} \tag{22}$$

assuming that the updated MPP $\mathbf{v}^{\mathrm{a}}_{\mathrm{k+1}}$ is located along the same radial direction v_N as the current MPP $\mathbf{v}^{*}_{\mathrm{k}}$ in V-space, as shown in Fig. 1. The updated MPP obtained from Eq. (22) is called the DRM-based MPP and denoted as $\mathbf{x}^{*}_{\mathrm{DRM}}$ in X-space. This DRM-based MPP is used to check whether or not the optimum design satisfies the constraint. The location of the DRM-based MPP for a concave and convex function is shown in Figs. 1(a) and 1(b), respectively.



(a) Concave Function



(b) Convex Function

Figure 1. DRM-based MPP for Concave and Convex Functions

Similar to the FORM, using the DRM-based inverse reliability analysis, the RBDO formulation in Eq. (5) can be rewritten as

$$\begin{aligned} &\text{minimize} && \mathrm{Cost}(\mathbf{d}) \\ &\text{subject to} && G_i(\mathbf{x}^{*}_{\mathrm{DRM}}) \le 0,\ i = 1, \cdots, nc \\ & && \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U,\quad \mathbf{d} \in R^{ndv} \text{ and } \mathbf{X} \in R^{n} \end{aligned} \tag{23}$$

5. M1A1 TANK ROADARM EXAMPLE

The roadarm of M1A1 tank is used to demonstrate applicability of the copula and DRM-based RBDO. The roadarm is modeled using 1572 eight-node isoparametric finite elements (SOLID45) and four beam elements (BEAM44) of a commercial finite element code, as shown in Fig. 2, and is made of S4340 steel with Young's modulus E=3.0×10^7 psi and Poisson's ratio ν=0.3. The durability analysis of the roadarm is carried out using Durability and Reliability Analysis Workspace (DRAW) (CCAD, 1999a-b), to obtain the fatigue life contour as shown in Fig. 3. The fatigue lives at the critical nodes shown in Fig. 3 are chosen as the design constraints of RBDO.

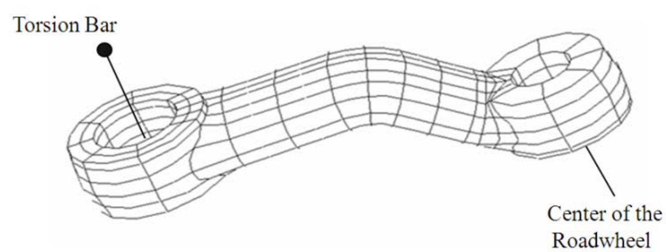


Figure 2. Finite Element Model of Roadarm

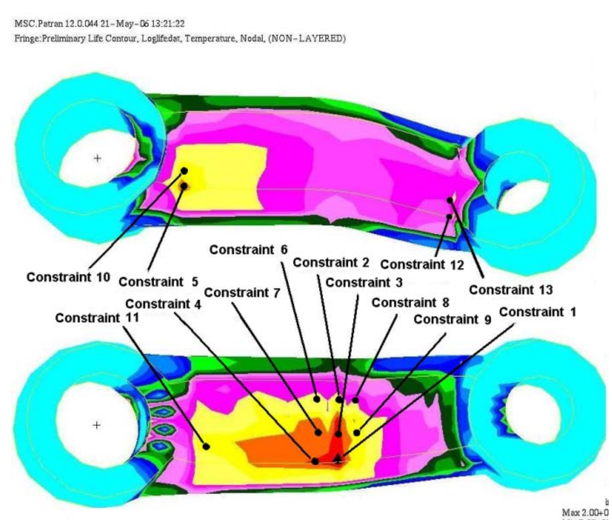


Figure 3. Fatigue Life Contour and Critical Nodes of Roadarm

The shape design variables are shown in Fig. 4. Eight shape design variables characterize four cross sectional shapes of the roadarm. Widths (x_1-direction) of the cross-sectional shapes are defined by the design variables d_1, d_3, d_5, and d_7 at the intersections 1 to 4, respectively, and heights (x_3-direction) of the cross sectional shapes are defined using the remaining four design variables. Eight shape design random variables are listed in Table 3 and assumed to be independent.

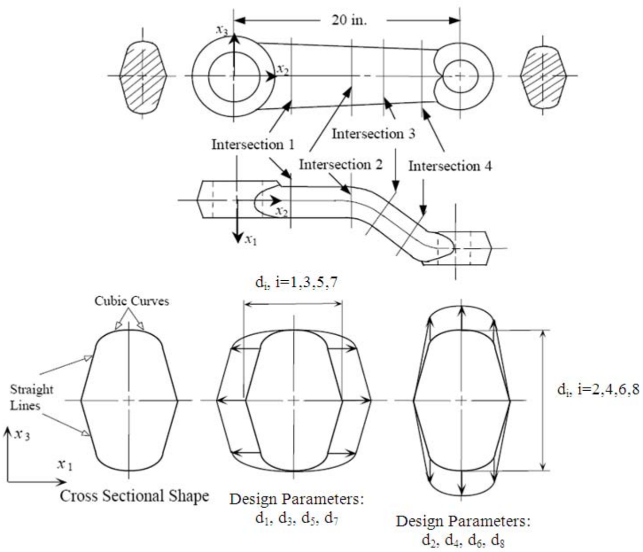


Figure 4. Shape Design Variables for Roadarm

Table 3. Properties of Input Geometry Random Variable

	$\mathbf{d}^L$	$\mathbf{d}^0$	$\mathbf{d}^U$	COV	Distr. Type
d_1	1.3500	1.7500	2.1500		Normal
d_2	2.6496	3.2496	3.7496		Normal
d_3	1.3500	1.7500	2.1500		Normal
d_4	2.5703	3.1703	3.6703	5%	Normal
d_5	1.3563	1.7563	2.1563		Normal
d_6	2.4377	3.0377	3.5377		Normal
d_7	1.3517	1.7517	2.1517		Normal
d_8	2.5085	2.9085	3.4085		Normal

For the input fatigue material properties, since statistical information of S4340 steel except its nominal value is not available, statistical information from 950X steel (Socie, 2003) is used to describe the properties of S4340 steel. Strain-Life relationship is usually given by the classical Coffin-Manson equation as (Meggiolaro and Castro, 2004)

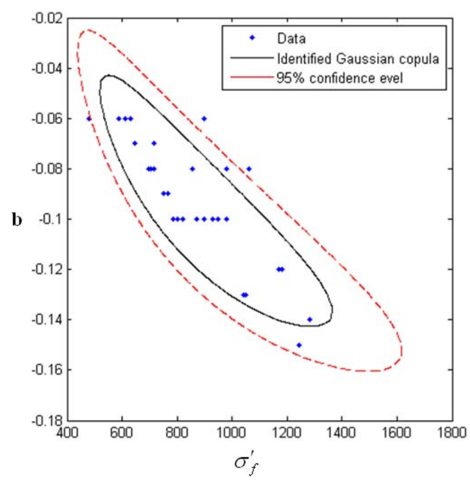
$$\frac{\Delta\varepsilon}{2} = \frac{\Delta\varepsilon_e}{2} + \frac{\Delta\varepsilon_p}{2} = \frac{\sigma_f'}{E}(2N_f)^b + \varepsilon_f'(2N_f)^c = \frac{\Delta\sigma}{2E} + \left(\frac{\Delta\sigma}{2K'}\right)^{1/n'} \tag{24}$$

where σ_f', b are the fatigue strength coefficient and exponent, ε_f' and c are the fatigue ductility coefficient and exponent, N_f is the fatigue initiation life, E is the Young's modulus, and K' and n' are the cyclic strength coefficient and exponent.

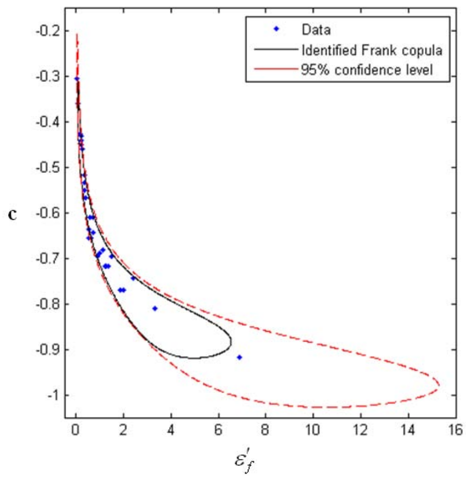
From Socie's study on 950X steel, it is shown that σ_f' and b have highly negative correlation with ρ=−0.828, and ε_f' and c have also highly negative correlation with ρ=−0.976. Using Bayesian method (Noh et al., 2008), Gaussian copula is identified for σ_f' and b, and Frank copula is identified for ε_f' and c as shown in Figs. 5(a) and 5(b). The fatigue material properties are listed in Table 4. In Table 4, COV is also obtained from Socie's study on 950X steel since COV for S4340 steel is not available in the literature. K' and n' in Table 4 are assumed to be independent.

Table 4. Properties of Input Fatigue Random Parameters

Non-design Uncertainties	Mean	COV	Distribution Type
K'	197000	25%	Lognormal
n'	0.1200	25%	Lognormal
σ_f'	177000	25%	Lognormal
b	-0.0730	25%	Normal
ε_f'	0.4100	50%	Lognormal
c	-0.6000	25%	Normal



(a) Gaussian Copula



(b) Frank Copula

Figure 5. Copulas for Fatigue Material Properties

The RBDO problem for the roadarm is formulated to

$$\begin{aligned} &\text{minimize} && \mathrm{Cost}(\mathbf{d}) \\ &\text{subject to} && P(G_i(\mathbf{d})>0) \le P_{F_i}^{\mathrm{Tar}}, \quad i=1,\cdots,nc \\ & && \mathbf{d}^{\mathrm{L}} \le \mathbf{d} \le \mathbf{d}^{\mathrm{U}} \end{aligned} \tag{25}$$

where

$$\begin{aligned} &\mathrm{Cost}(\mathbf{d}): \text{Weight of Roadarm} \\ &G_i(\mathbf{d}) = 1 - \frac{L(\mathbf{d})}{L_t}, \; i=1,\cdots,nc \\ &L(\mathbf{d}): \text{Crack Initiation Fatigue Life,} \\ &L_t: \text{Crack Initiation Target Fatigue Life (=5 years)} \\ &P_{F_i}^{\mathrm{Tar}} = 2.275\% \end{aligned} \tag{26}$$

and number of constraints nc = 13 as shown in Fig. 3

First, we assume that the input random variables are independent and run the RBDO. Second, we use correlation of input fatigue material properties to test the applicability of the copula but run the FORM-based RBDO. Finally, the DRM-based RBDO with correlated input variables is carried out to obtain more accurate optimum.

The RBDO test results for each case are shown in Table 5. Interestingly, the FORM-based RBDO assuming that input variables are independent is failed because the feasible region in U-space is small, and thus, there is no feasible solution within the design bounds. However, the FORM-based RBDO with two correlated pairs converges to the optimum in Table 5 and shows significant reduction in the weight of the roadarm compared to the independent case. This is because the high correlations make feasible region much larger than the independent case.

Table 5. RBDO Comparison

	Initial	D.O.*	FORM		DRM
			Indep.	Correlated	
d_1	1.750	1.588	2.117	1.958	1.928
d_2	3.250	2.650	3.427	2.650	2.650
d_3	1.750	1.922	2.044	2.031	2.067
d_4	3.170	2.570	3.670	2.670	2.577
d_5	1.756	1.477	1.939	1.775	1.776
d_6	3.038	3.292	3.538	3.538	3.535
d_7	1.752	1.630	2.152	2.152	2.075
d_8	2.908	2.508	3.408	2.536	2.512
Cost	515.09	464.56	617.38	519.70	514.02
Active Const.	Infeasible	1,3,5, 8,12	**Failed**	1,3,5, 9,13	1,5,9

* D.O. means deterministic optimum.

At the optimum of the FORM-based RBDO with correlated input variables, reliability analysis using the MPP-based DRM is carried out to check whether or not the constraint functions are linear. Table 6 shows the reliability analysis results for active constraints. As shown in Table 6, the probabilities of failure for active constraints 1, 3, 9 are less than the target probability of failure (2.275%), which means that the optimum design obtained from the FORM-based RBDO is still conservative. Hence, using the DRM-based RBDO, the weight of the roadarm can be further reduced while satisfying the target probability of failure. From Table 5, it can be seen that the weight of roadarm is further decreased from 519.70 to 514.02 by use of the DRM-based RBDO.

Table 6. Reliability Analysis Results Using DRM

	G_1	G_3	G_5	G_9	G_{13}
P_F, %	1.588	1.722	2.313	1.679	2.240

CONCLUSIONS

In this paper, copulas are proposed to model the joint CDFs using marginal CDFs and samples without requiring effort to directly capture the joint CDFs. Due to the nonlinear Rosenblatt transformation caused by non-Gaussian copula, constraint functions can be highly nonlinear in U-space. For more accurate inverse reliability analysis method than FORM, the MPP-based DRM is also proposed to handle the nonlinearity or multi-dimensionality of the problem. The roadarm of an M1A1 tank is used to demonstrate the applicability of the copula and MPP-based DRM for RBDO. The numerical test using the roadarm shows that using the copula to improve fidelity of the input joint CDF, the weight of the roadarm is significantly reduced compared to the RBDO with independent variables. However, due to the inaccuracy of FORM for the calculation of the probability of failure, the optimum is still conservative than the FORM result in Table 5. By using the DRM-based RBDO, the weight of the roadarm can be further reduced while satisfying the target probability of failure. In conclusion, the RBDO can be significantly improved by using copulas to obtain a better estimation of the joint CDFs

and MPP-based DRM to obtain a better calculation of the probability of failure.

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